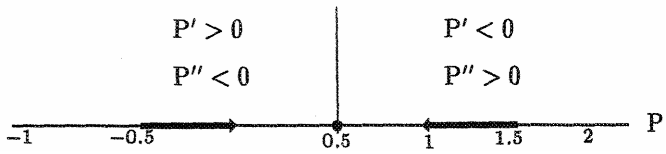
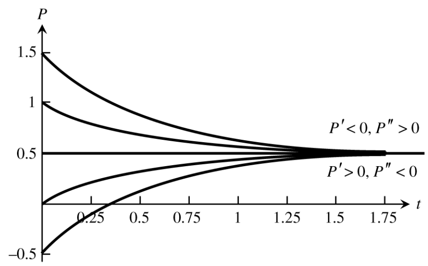
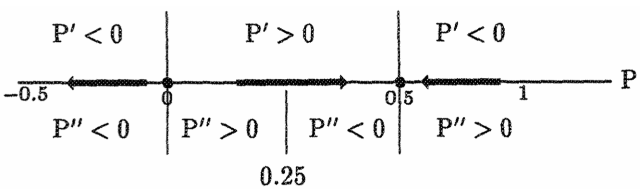
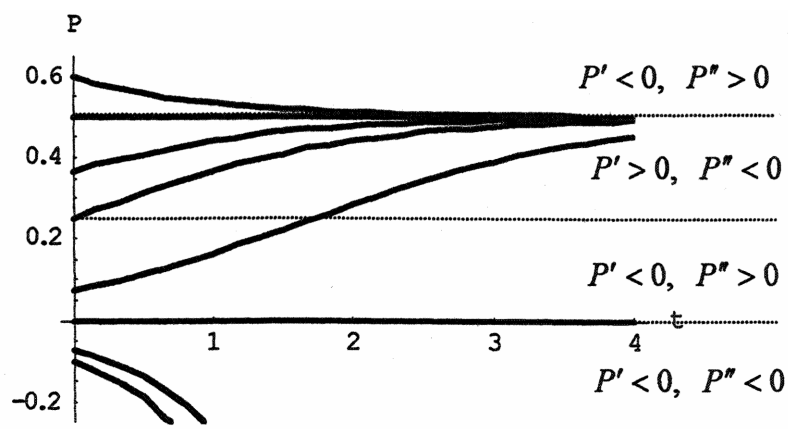
9. $\frac{dP}{dt} = 1 - 2P$ has a stable equilibrium at $P = \frac{1}{2}$. $\frac{d^2P}{dt^2} = -2\frac{dP}{dt} = -2(1 - 2P)$



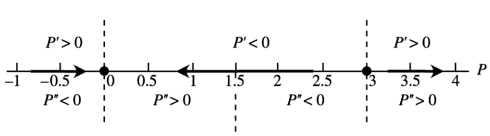


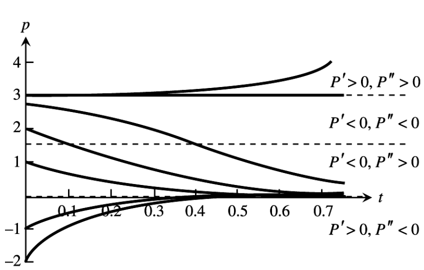
10. $\frac{dP}{dt} = P(1 - 2P)$ has an unstable equilibrium at $P = 0$ and a stable equilibrium at $P = \frac{1}{2}$.
$\frac{d^2P}{dt^2} = (1 - 4P)\frac{dP}{dt} = P(1 - 4P)(1 - 2P)$



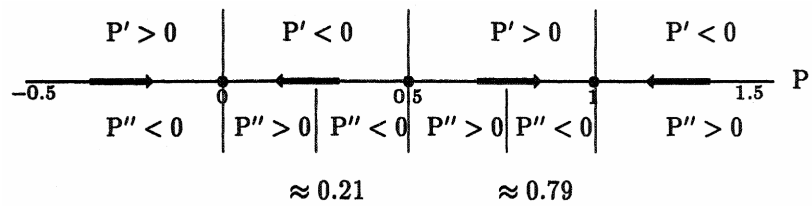


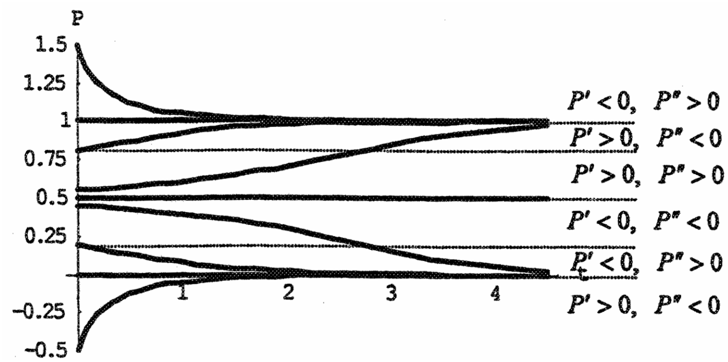
11. $\frac{dP}{dt} = 2P(P - 3)$ has a stable equilibrium at $P = 0$ and an unstable equilibrium at $P = 3$.
$\frac{d^2P}{dt^2} = 2(2P - 3)\frac{dP}{dt} = 4P(2P - 3)(P - 3)$



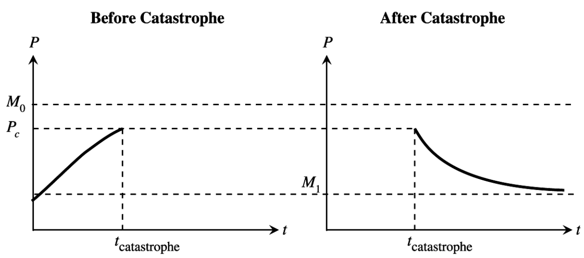


12. $\frac{dP}{dt} = 3P(1-P)\left(P-\frac{1}{2}\right)$ has a stable equilibria at $P=0$ and $P=1$ an unstable equilibrium at $P=\frac{1}{2}$.
$\frac{d^2P}{dt^2} = -\frac{3}{2}(6P^2-6P+1)\frac{dP}{dt} = \frac{3}{2}P\left(P-\frac{3-\sqrt{3}}{6}\right)\left(P-\frac{1}{2}\right)\left(P-\frac{3+\sqrt{3}}{6}\right)(P-1)$



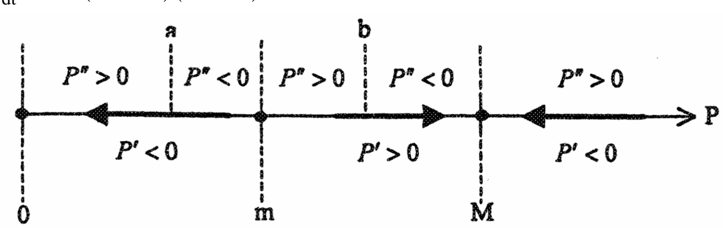


13.



Before the catastrophe, the population exhibits logistic growth and $P(t) \to M_0$, the stable equilibrium. After the catastrophe, the population declines logistically and $P(t) \to M_1$, the new stable equilibrium.

14. $\frac{dP}{dt} = rP(M-P)(P-m)$, $r, M, m > 0$



The model has 3 equilibrium points. The rest point $P=0$, $P=M$ are asymptotically stable while $P=m$ is unstable. For initial populations greater than m, the model predicts P approaches M for large t. For initial populations less than m, the model predicts extinction. Points of inflection occur at $P=a$ and $P=b$ where $a = \frac{1}{3}\left[M+m-\sqrt{M^2-mM+m^2}\right]$ and $b = \frac{1}{3}\left[M+m+\sqrt{M^2-mM+m^2}\right]$.

(a) The model is reasonable in the sense that if $P < m$, then $P \to 0$ as $t \to \infty$; if $m < P < M$, then $P \to M$ as $t \to \infty$; if $P > M$, then $P \to M$ as $t \to \infty$.

(b) It is different if the population falls below m, for then $P \to 0$ as $t \to \infty$ (extinction). If is probably a more realistic model for that reason because we know some populations have become extinct after the population level became too low.

(c) For $P > M$ we see that $\frac{dP}{dt} = rP(M - P)(P - m)$ is negative. Thus the curve is everywhere decreasing. Moreover, $P \equiv M$ is a solution to the differential equation. Since the equation satisfies the existence and uniqueness conditions, solution trajectories cannot cross. Thus, $P \to M$ as $t \to \infty$.

(d) See the initial discussion above.

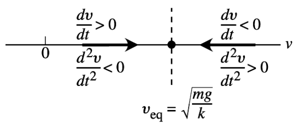
(e) See the initial discussion above.

15. $\frac{dv}{dt} = g - \frac{k}{m}v^2$, g, k, $m > 0$ and $v(t) \ge 0$

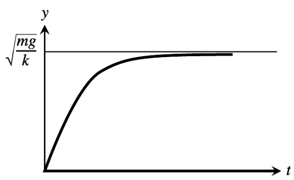
Equilibrium: $\frac{dv}{dt} = g - \frac{k}{m}v^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$

Concavity: $\frac{d^2v}{dt^2} = -2\left(\frac{k}{m}v\right)\frac{dv}{dt} = -2\left(\frac{k}{m}v\right)\left(g - \frac{k}{m}v^2\right)$

(a)



(b)



(c) $v_{terminal} = \sqrt{\frac{160}{0.005}} = 178.9 \frac{ft}{s} = 122$ mph

16. $F = F_p - F_r$

$ma = mg - k\sqrt{v}$

$\frac{dv}{dt} = g - \frac{k}{m}\sqrt{v}$, $v(0) = v_0$

Thus, $\frac{dv}{dt} = 0$ implies $v = \left(\frac{mg}{k}\right)^2$, the terminal velocity. If $v_0 < \left(\frac{mg}{k}\right)^2$, the object will fall faster and faster, approaching the terminal velocity; if $v_0 > \left(\frac{mg}{k}\right)^2$, the object will slow down to the terminal velocity.

17. $F = F_p - F_r$

$ma = 50 - 5|v|$

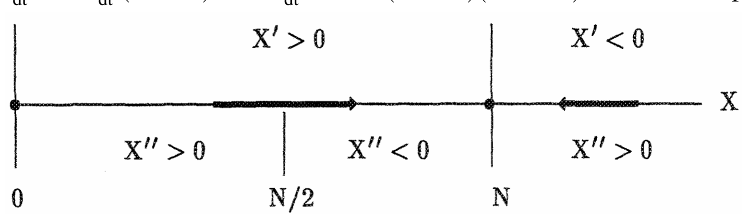
$\frac{dv}{dt} = \frac{1}{m}(50 - 5|v|)$

The maximum velocity occurs when $\frac{dv}{dt} = 0$ or $v = 10 \frac{ft}{sec}$.

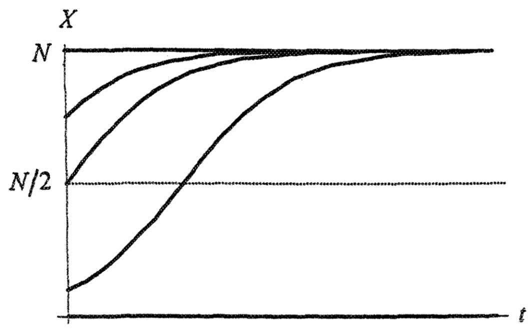
18. (a) The model seems reasonable because the rate of spread of a piece of information, an innovation, or a cultural fad is proportional to the product of the number of individuals who have it (X) and those who do not $(N - X)$. When X is small, there are only a few individuals to spread the item so the rate of spread is slow. On the other hand, when $(N - X)$ is small the rate of spread will be slow because there are only a few indicidual who can receive it during the interval of time. The rate of spread will be fastest when both X and $(N - X)$ are large because then there are a lot of individuals to spread the item and a lot of individuals to receive it.

(b) There is a stable equilibrium at $X = N$ and an unstable equilibrium at $X = 0$.

$\frac{d^2X}{dt^2} = k\frac{dX}{dt}(N - X) - kX\frac{dX}{dt} = k^2X(N - X)(N - 2X) \Rightarrow$ inflection points at $X = 0$, $X = \frac{N}{2}$, and $X = N$.



(c)



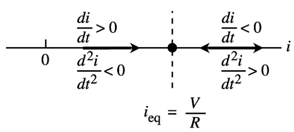
(d) The spread rate is most rapid when $x = \frac{N}{2}$. Eventually all of the people will receive the item.

19. $L\frac{di}{dt} + Ri = V \Rightarrow \frac{di}{dt} = \frac{V}{L} - \frac{R}{L}i = \frac{R}{L}\left(\frac{V}{R} - i\right)$, V, L, R > 0

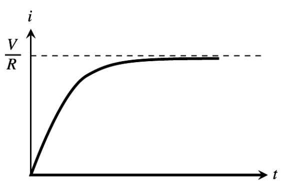
Equilibrium: $\frac{di}{dt} = \frac{R}{L}\left(\frac{V}{R} - i\right) = 0 \Rightarrow i = \frac{V}{R}$

Concavity: $\frac{d^2i}{dt^2} = -\left(\frac{R}{L}\right)\frac{di}{dt} = -\left(\frac{R}{L}\right)^2\left(\frac{V}{R} - i\right)$

Phase Line:

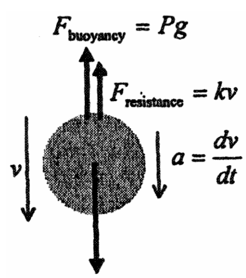


If the switch is closed at $t = 0$, then $i(0) = 0$, and the graph of the solution looks like this:



As $t \to \infty$, it $\to i_{\text{steady state}} = \frac{V}{R}$. (In the steady state condition, the self-inductance acts like a simple wire connector and, as a result, the current throught the resistor can be calculated using the familiar version of Ohm's Law.)

20. (a) Free body diagram of the pearl:

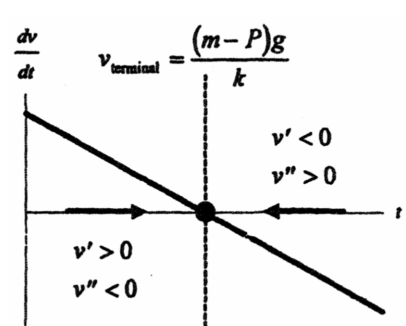


(b) Use Newton's Second Law, summing forces in the direction of the acceleration:
$mg - Pg - kv = ma \Rightarrow \frac{dv}{dt} = \left(\frac{m-P}{m}\right)g - \frac{k}{m}v.$

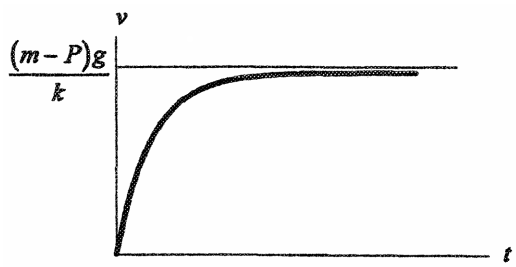
(c) Equilibrium: $\frac{dv}{dt} = \frac{k}{m}\left(\frac{(m-P)g}{k} - v\right) = 0$

$\Rightarrow v_{\text{terminal}} = \frac{(m-P)g}{k}$

Concavity: $\frac{d^2v}{dt^2} = -\frac{k}{m}\frac{dv}{dt} = -\left(\frac{k}{m}\right)^2\left(\frac{(m-P)g}{k} - v\right)$



(d)



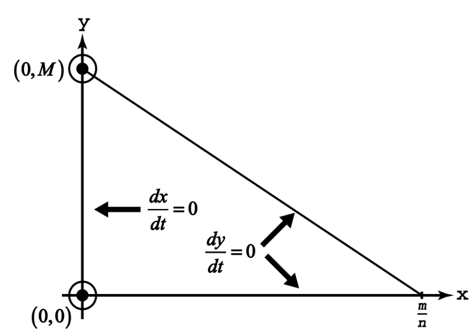
(e) The terminal velocity of the pearl is $\frac{(m-P)g}{k}$.

9.5 SYSTEMS OF EQUATIONS AND PHASE PLANES

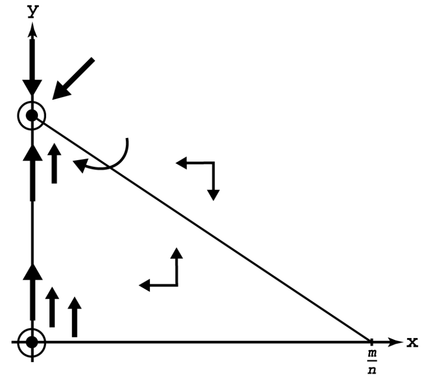
1. Seasonal variations, nonconformity of the environments, effects of other interactions, unexpected disasters, etc.

2. $x = r\cos\theta \Rightarrow \frac{dx}{dt} = -r\sin\theta\,\frac{d\theta}{dt} + \cos\theta\,\frac{dr}{dt} = y + x - x(x^2+y^2) = r\sin\theta + r\cos\theta - r^3\cos\theta$
$y = r\sin\theta \Rightarrow \frac{dy}{dt} = r\cos\theta\,\frac{d\theta}{dt} + \sin\theta\,\frac{dr}{dt} = -x + y - x(x^2+y^2) = -r\cos\theta + r\sin\theta - r^3\sin\theta$
Solve for $\frac{dr}{dt}$ by adding $\cos\theta \times$ eq(1) to $\sin\theta \times$ eq(2):
$\cos^2\theta\,\frac{dr}{dt} + \sin^2\theta\,\frac{dr}{dt} = \cos\theta(r\sin\theta + r\cos\theta - r^3\cos\theta) + \sin\theta(-r\cos\theta + r\sin\theta - r^3\sin\theta)$
$\Rightarrow \frac{dr}{dt} = r\sin\theta\cos\theta + r\cos^2\theta - r^3\cos^2\theta - r\sin\theta\cos\theta + r\sin\theta - r^3\sin^2\theta = r - r^3 = r(1-r^2)$
Solve for $\frac{d\theta}{dt}$ by adding $(-\sin\theta) \times$ eq(1) to $\cos\theta \times$ eq(2):
$r\sin^2\theta\frac{d\theta}{dt} + r\cos^2\theta\,\frac{d\theta}{dt} = -\sin\theta(r\sin\theta + r\cos\theta - r^3\cos\theta) + \cos\theta(-r\cos\theta + r\sin\theta - r^3\sin\theta)$
$\Rightarrow r\frac{d\theta}{dt} = -r\sin^2\theta - r\sin\theta\cos\theta + r^3\sin\theta\cos\theta - r\cos^2\theta + r\sin\theta\cos\theta - r^3\sin\theta\cos\theta = -r \Rightarrow \frac{d\theta}{dt} = -1$
If $r = 1$ (that is, the trajectory starts on the circle $x^2 + y^2 = 1$), then $\left.\frac{dr}{dt}\right|_{r=1} = (1)\left(1 - (1)^2\right) = 0$, thus the trajectory remains on the circle, and rotates around the circle in a clockwise direction, since $\frac{d\theta}{dt} = -1$. The solution is periodic since at any point (x, y) on the trajectory, $(x, y) = (r\cos\theta, r\sin\theta) = (1\cos\theta, 1\sin\theta) = (\cos\theta, \sin\theta) \Rightarrow$ both x and y are periodic.

3. This model assumes that the number of interactions is porportional to the product of x and y:
$\frac{dx}{dt} = (a - by)x,\ a < 0,\ \frac{dy}{dt} = m\left(1 - \frac{y}{M}\right)y - nxy = y\left(m - \frac{m}{M}y - nx\right)$.
To find the equilibrium points:
$\frac{dx}{dt} = 0 \Rightarrow (a - by)x = 0 \Rightarrow x = 0$ or $y = \frac{a}{b}$
(remember $\frac{a}{b} < 0$);
$\frac{dy}{dt} = 0 \Rightarrow y\left(m - \frac{m}{M}y - nx\right) \Rightarrow y = 0$ or $y = -\frac{Mn}{m}x + M$;
Thus there are two equlibrium points, both occur when $x = 0$, $(0, 0)$ and $(0, M)$.



Implies coexistence is not possible because eventually trout die out and bass reach their population limit.



4. The coefficients a, b, m, and n need to be determined by sampling or by analyzing historical data. Then, more specific graphical predictions can be made. These predictions would then have to be compared to actual population growth patterns. If the predictions match actual results, we have partially validated our model. If necessary, more tests could be run. However, it should be remembered that the primary purpose of a graphical analysis is to analyze the behavior qualitatively. With reference to Figure 9.29, attempt to maintain the fish populations in Region B through stocking and regulation (open and closed seasons). For example, should Regions A or D be entered, restocking the appropriate species can cause a return to Region B.

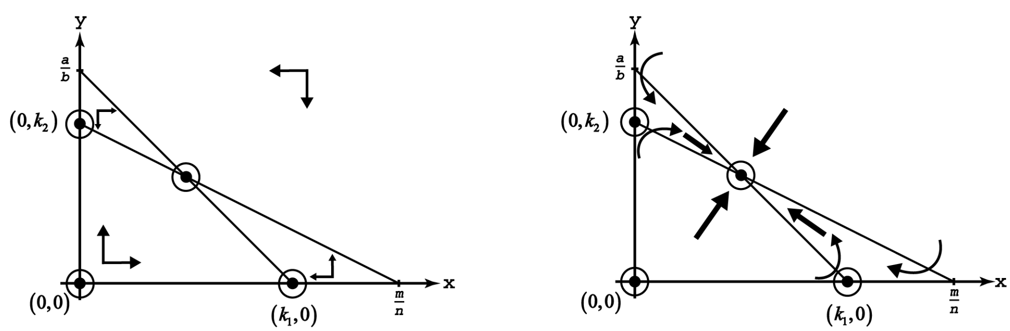
5. (a) Logistic growth occurs in the absence of the competitor, and simple interaction of the species: growth dominates the competition when either population is small so it is difficult to drive either species to extinction.

   (b) $a =$ per capita growth rate for trout
   $m =$ per capita growth rate for bass
   $b =$ intensity of competition to the trout
   $n =$ intensity of competition to the bass
   $k_1 =$ environmental carrying capacity for the trout
   $k_2 =$ environmental carrying capacity for the bass

   (c) $\frac{dx}{dt} = 0 \Rightarrow a\left(1 - \frac{x}{k_1}\right)x - bxy = \left[a\left(1 - \frac{x}{k_1}\right) - by\right]x = 0 \Rightarrow x = 0$ or $a\left(1 - \frac{x}{k_1}\right) - by = 0 \Rightarrow x = 0$ or $y = \frac{a}{b} - \frac{a}{bk_1}x$; $\frac{dy}{dt} = 0 \Rightarrow m\left(1 - \frac{y}{k_2}\right)y - nxy = \left[m\left(1 - \frac{y}{k_2}\right) - nx\right]y = 0 \Rightarrow y = 0$ or $m\left(1 - \frac{y}{k_2}\right) - nx = 0 \Rightarrow y = 0$ or $y = k_2 - \frac{nk_2}{m}x$. There are five cases to consider.

   Case I: $\frac{a}{b} > k_2$ and $\frac{m}{n} > k_1$.

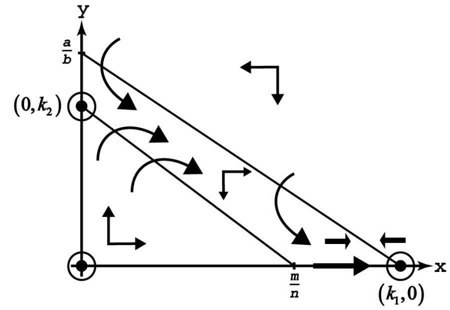
   By picking $\frac{a}{b} > k_2$ and $\frac{m}{n} > k_1$ we ensure an equilibrium point exists inside the first quadrant.



   Graphical analysis implies four equilibrium points exist: $(0, 0)$, $(k_1, 0)$, $(0, k_2)$, and $\left(\frac{amk_1 - bmk_1k_2}{am - bnk_1k_2}, \frac{amk_2 - ank_1k_2}{am - bnk_1k_2}\right)$ (the point of intersection of the two boundaries in the first quadrant). All of these equilibrium points are unstable except for the point of intersection. The possibility of coexistence is predicted by this model.

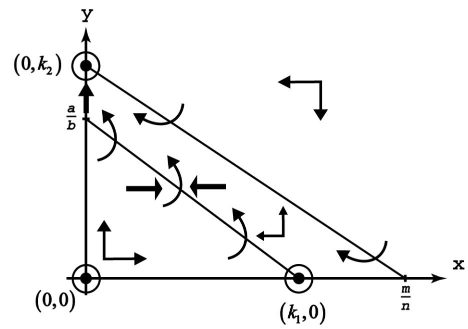
Case II: $\frac{a}{b} > k_2$ and $\frac{m}{n} < k_1$.

$(0, k_2)$: unstable
$(k_1, 0)$: stable
$(0, 0)$: unstable
Trout wins: $(k_1, 0)$
Not sensitive
No coexistence



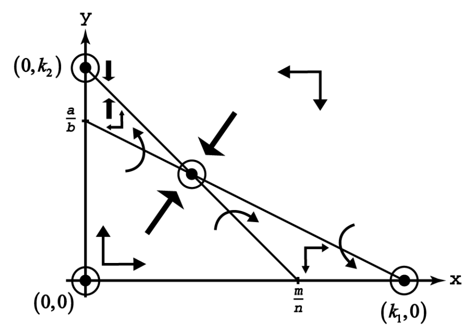
Case III: $\frac{a}{b} < k_2$ and $\frac{m}{n} > k_1$.

$(0, k_2)$: stable
$(k_1, 0)$: unstable
$(0, 0)$: unstable
Bass wins: $(0, k_2)$
Not sensitive
No coexistence



Case IV: $\frac{a}{b} < k_2$ and $\frac{m}{n} < k_1$.

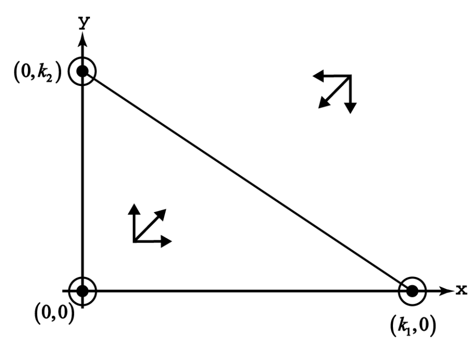
$(0, k_2)$: stable
$(k_1, 0)$: stable
$(0, 0)$: unstable
$\left(\frac{amk_1 - bmk_1k_2}{am - bnk_1k_2}, \frac{amk_2 - ank_1k_2}{am - bnk_1k_2}\right)$: unstable
Bass or trout: $(0, k_2)$ or $(k_1, 0)$
Very sensitive
Coexistence is <u>possible</u> but not predicted



If we assume $\frac{a}{b} < k_2$ and $\frac{m}{n} < k_1$ then graphical analysis implies four equilibrium poins exist: $(0, k_2)$, $(k_1, 0)$, $(0, 0)$, and $\left(\frac{amk_1 - bmk_1k_2}{am - bnk_1k_2}, \frac{amk_2 - ank_1k_2}{am - bnk_1k_2}\right)$ (the point of intersection of the two boundaries in the first quadrant).

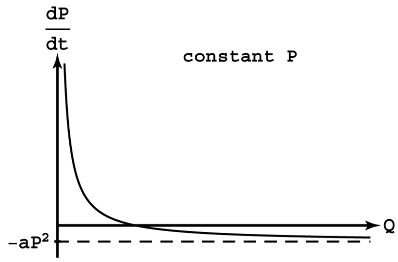
Case V: $\frac{a}{b} = k_2$ and $\frac{a}{bk_1} = \frac{nk_2}{m}$ (lines coincide).

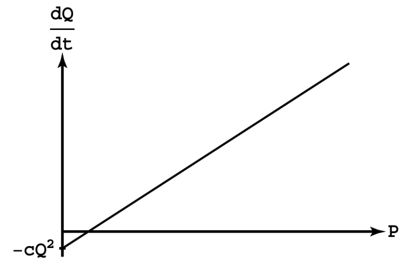
$(0, k_2)$: stable
$(k_1, 0)$: stable
$(0, 0)$: unstable
Line segment joining $(0, k_2)$ and $(k_1, 0)$: stable
Bass wins: $(0, k_2)$
Not sensitive
Coexistence is likely outcome



Note that all points on the line segment joining $(0, k_2)$ and $(k_1, 0)$ are rest points.

6. For a fixed price, as Q increases, $\frac{dP}{dt}$ gest smaller and, possibly, becomes negative. This observation implies that as the quantity supplied increases, the price will not rise as fast. If Q gets high enough, then the price will decrease. Next, consider $\frac{dQ}{dt}$: For a fixed quantity, as P increases, $\frac{dQ}{dt}$ gets larger. Thus, as the market price increases, the quantity supplied will increase at a faster rate. If P is too small, $\frac{dQ}{dt}$ will be negative and the quantity supplied will decrease. This observation is the traditional explanation of the effect of market price levels on the quantity supplied.

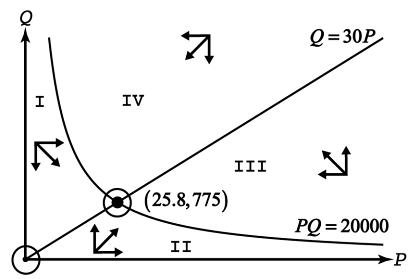




(a) $\frac{dP}{dt} = 0$ and $\frac{dQ}{dt} = 0$ gives the equilibrium points (P, Q): (0, 0) and (25.8, 775).
Now $\frac{dP}{dt} > 0$ when $PQ < 20{,}000$ and $P > 0$; $\frac{dP}{dt} < 0$ otherwise. $\frac{dQ}{dt} > 0$ when $P > \frac{Q}{30}$ and $Q > 0$; $\frac{dQ}{dt} < 0$ otherwise.

(b) These considerations give the following graphical analysis:

Region	$\frac{dP}{dt}$	$\frac{dQ}{dt}$
I	> 0	< 0
II	> 0	> 0
III	< 0	< 0
IV	< 0	> 0



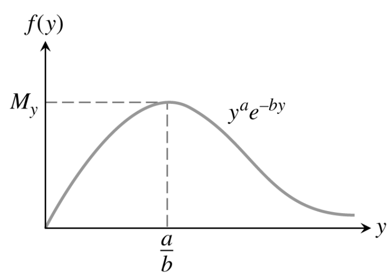
The equilibrium point (0, 0) is unstable. The graphical analysis for the point (25.8, 775) is inconclusive: trajectories near the point may be periodic, or may spiral toward or away from the point.

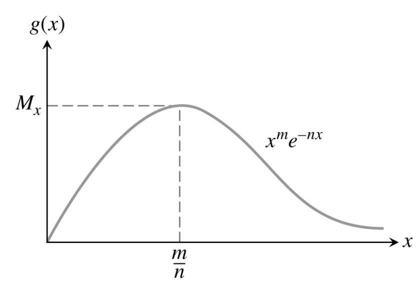
(c) The curve $\frac{dP}{dt} = 0$ or $PQ = 20000$ can be thought of as the demand curve; $\frac{dQ}{dt} = 0$ or $Q = 30P$ can be viewed as the supply curve.

7. (a) $\frac{dx}{dt} = ax - bxy = (a - by)x$ and $\frac{dy}{dt} = my - nxy = (m - nx)y \Rightarrow \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{(m-nx)y}{(a-by)x}$

(b) $\frac{dy}{dx} = \frac{(m-nx)y}{(a-by)x} \Rightarrow \left(\frac{a}{y} - b\right)dy = \left(\frac{m}{x} - n\right)dx \Rightarrow \int\left(\frac{a}{y} - b\right)dy = \int\left(\frac{m}{x} - n\right)dx \Rightarrow a\ln|y| - by = m\ln|x| - nx + C$
$\Rightarrow \ln|y^a| + \ln e^{-by} = \ln|x^m| + \ln e^{-nx} + \ln e^C \Rightarrow \ln|y^a e^{-by}| = \ln|x^m e^{-nx} e^C| \Rightarrow y^a e^{-by} = x^m e^{-nx} e^C$, let $K = e^C$
$\Rightarrow y^a e^{-by} = Kx^m e^{-nx}$

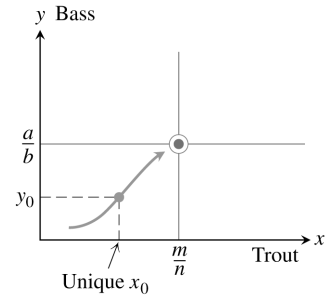
(c) $f(y) = y^a e^{-by} \Rightarrow f'(y) = ay^{a-1}e^{-by} - by^a e^{-by} = y^{a-1}e^{-by}(a - by)$ and $f'(y) = 0 \Rightarrow y = 0$ or $y = \frac{a}{b}$;
$f''\left(\frac{a}{b}\right) = -b\left(\frac{a}{b}\right)^{a-1}e^{-a} < 0 \Rightarrow f(y)$ has a unique max of $M_y = \left(\frac{a}{eb}\right)^a$ when $y = \frac{a}{b}$. $g(x) = x^m e^{-nx}$
$\Rightarrow g'(x) = mx^{m-1}e^{-nx} - nx^m e^{-nx} = x^{m-1}e^{-nx}(m - nx)$ and $g'(x) = 0 \Rightarrow x = 0$ or $x = \frac{m}{n}$;
$g''\left(\frac{m}{n}\right) = -n\left(\frac{m}{n}\right)^{m-1}e^{-m} < 0 \Rightarrow g(x)$ has a unique max of $M_x = \left(\frac{m}{en}\right)^m$ when $x = \frac{m}{n}$.





(d) Consider trajectory $(x, y) \to \left(\frac{m}{n}, \frac{a}{b}\right)$. $y^a e^{-by} = Kx^m e^{-nx} \Rightarrow \frac{y^a}{e^{by}} \cdot \frac{e^{nx}}{x^m} = K$, taking the limit of both sides
$\Rightarrow \lim\limits_{\substack{x \to m/n \\ y \to a/b}} \left(\frac{y^a}{e^{by}} \cdot \frac{e^{nx}}{x^m}\right) = \lim\limits_{\substack{x \to m/n \\ y \to a/b}} K \Rightarrow \frac{M_y}{M_x} = K$. Thus, $\frac{y^a}{e^{by}} = \frac{M_y}{M_x}\frac{x^m}{e^{nx}}$ represents the equation any solution trajectory must satisfy if the trajectory approaches the rest point asymptotically.

(e) Pick initial condition $y_0 < \frac{a}{b}$. Then, from the figure at right, $f(y_0) < M_y$ implies $\frac{M_y}{M_x}\frac{x^m}{e^{nx}} = \frac{y_0^a}{e^{b y_0}} < M_y$ and thus $\frac{x^m}{e^{nx}} < M_x$. From the figure for $g(x)$, there exists a unique $x_0 < \frac{m}{n}$ satisfying $\frac{x^m}{e^{nx}} < M_x$. That is, for each $y < \frac{a}{b}$ there is a unique x satisfying $\frac{y^a}{e^{by}} = \frac{M_y}{M_x}\frac{x^m}{e^{nx}}$. Thus, there can exist only one trajectory solution approaching $\left(\frac{m}{n}, \frac{a}{b}\right)$. (You can think of the point (x_0, y_0) as the initial condition for that trajectory.)



(f) Likewise there exists a unique trajectory when $y_0 > \frac{a}{b}$. Again, $f(y_0) < M_y$ implies $\frac{M_y}{M_x}\frac{x^m}{e^{nx}} = \frac{y_0^a}{e^{b y_0}} < M_y$ and thus $\frac{x^m}{e^{nx}} < M_x$. From the figure for $g(x)$, there exists a unique $x_0 > \frac{m}{n}$ satisfying $\frac{x^m}{e^{nx}} < M_x$. That is, for each $y > \frac{a}{b}$ there is a unique x satisfying $\frac{y^a}{e^{by}} = \frac{M_y}{M_x}\frac{x^m}{e^{nx}}$. Thus, there can exist only one trajectory solution approaching $\left(\frac{m}{n}, \frac{a}{b}\right)$.

8. Let $z = y' = \frac{dy}{dx} \Rightarrow \frac{dz}{dx} = z' = y''$, then given the differential equation $y'' = F(x, y, y')$, we can write it as the following system of first order differential equations:
$$\frac{dy}{dx} = z$$
$$\frac{dz}{dx} = F(x, y, z)$$
In general, for the n^{th} order differential equation given by $y^{(n)} = F\left(x, y, y', y'', \ldots, y^{(n-1)}\right)$, let $z_1 = y' = \frac{dy}{dx}$ $\Rightarrow \frac{dz_1}{dx} = z_1' = y''$, let $z_2 = z_1' = y''$, $\Rightarrow \frac{dz_2}{dx} = z_2' = y'''$, ..., let $z_{n-1} = z_{n-2}' = y^{(n-1)} \Rightarrow z_{n-1}' = y^{(n)}$. This gives us the following system of first order differential equations:
$$\frac{dy}{dx} = z_1$$
$$\frac{dz_1}{dx} = z_2$$
$$\frac{dz_2}{dx} = z_3$$
$$\vdots$$
$$\frac{dz_{n-2}}{dx} = z_{n-1}$$
$$\frac{dz_{n-1}}{dx} = F(x, y, z_1, z_2, \ldots, z_{n-1})$$

9. In the absence of foxes $\Rightarrow b = 0 \Rightarrow \frac{dx}{dt} = a\,x$ and the population of rabbits grows at a rate proportional to the number of rabbits.

10. In the absence of rabbits $\Rightarrow d = 0 \Rightarrow \frac{dy}{dt} = -c\,y$ and the population of foxes decays (since the foxes have no food source) at a rate proportional to the number of foxes.

11. $\frac{dx}{dt} = (a - b\,y)x = 0 \Rightarrow y = \frac{a}{b}$ or $x = 0$; $\frac{dy}{dt} = (-c + d\,x)y = 0 \Rightarrow x = \frac{c}{d}$ or $y = 0 \Rightarrow$ equilibrium points at $(0, 0)$ or $\left(\frac{c}{d}, \frac{a}{b}\right)$. For the point $(0, 0)$, there are no rabbits and no foxes. It is an unstable equilibrium point, if there are no foxes, but a few rabbits are introduced, then $\frac{dx}{dt} = a \Rightarrow$ the rabbit population will grow exponentially away from $(0, 0)$

12. Let $x(t)$ and $y(t)$ both be positive and suppose that they satisfy the differential equations $\frac{dx}{dt} = (a - b\,y)x$ and $\frac{dy}{dt} = (-c + d\,x)y$. Let $C(t) = a\ln y(t) - b\,y(t) - d\,x(t) + c\ln x(t) \Rightarrow C'(t) = a\,\frac{y'(t)}{y(t)} - b\,y'(t) - d\,x'(t) + c\,\frac{x'(t)}{x(t)}$
$= \left(\frac{a}{y(t)} - b\right)y'(t) + \left(\frac{c}{x(t)} - d\right)x'(t) = \left(\frac{a}{y(t)} - b\right)(-c + d\,x(t))x(t) + \left(\frac{c}{x(t)} - d\right)(a - b\,y(t))y(t) = 0$
Since $C'(t) = 0 \Rightarrow C(t) =$ constant.

13. Consider a particular trajectory and suppose that (x_0, y_0) is such that $x_0 < \frac{c}{d}$ and $y_0 < \frac{a}{b}$, then $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0 \Rightarrow$ the rabbit population is increasing while the fox population is decreasing, points on the trajectory are moving down and to the right; if $x_0 > \frac{c}{d}$ and $y_0 < \frac{a}{b}$, then $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0 \Rightarrow$ both the rabbit and fox populations are increasing, points on the trajectory are moving up and to the right; if $x_0 > \frac{c}{d}$ and $y_0 > \frac{a}{b}$, then $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0 \Rightarrow$ the rabbit population is decreasing while the fox population is increasing, points on the trajectory are moving up and to the left; and finally if

$x_0 < \frac{c}{d}$ and $y_0 > \frac{a}{b}$, then $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} < 0 \Rightarrow$ both the rabbit and fox populations are decreasing, points on the trajectory are moving down and to the left. Thus, points travel around the trajectory in a counterclockwise direction. Note that we will follow the same trajectory if (x_0, y_0) starts at a different point on the trajectory.

14. There are three possible cases: If the rabbit population begins (before the wolf) and ends (after the wolf) at a value larger than the equilibrium level of $x = \frac{c}{d}$, then the trajectory moves closer to the equilibrium and the maximum value of the foxes is smaller. If the rabbit population begins (before the wolf) and ends (after the wolf) at a value smaller than the equilibrium level of $x = \frac{c}{d}$, but greater than 0, then the trajectory moves further from the equilibrium and the maximum value of the foxes is greater. If the rabbit population begins and ends very near the equilibrium value, then the trajectory will stay near the equilibrium value, since it is a stable equilibrium, and the fox population will remain roughly the same.

CHAPTER 9 PRACTICE EXERCISES

1. $y' = xe^y\sqrt{x-2} \Rightarrow e^{-y}dy = x\sqrt{x-2}\,dx \Rightarrow -e^{-y} = \frac{2(x-2)^{3/2}(3x+4)}{15} + C \Rightarrow e^{-y} = \frac{-2(x-2)^{3/2}(3x+4)}{15} - C$
$\Rightarrow -y = \ln\left[\frac{-2(x-2)^{3/2}(3x+4)}{15} - C\right] \Rightarrow y = -\ln\left[\frac{-2(x-2)^{3/2}(3x+4)}{15} - C\right]$

2. $y' = xye^{x^2} \Rightarrow \frac{dy}{y} = e^{x^2}x\,dx \Rightarrow \ln y = \frac{1}{2}e^{x^2} + C$

3. $\sec x\,dy + x\cos^2 y\,dx = 0 \Rightarrow \frac{dy}{\cos^2 y} = -\frac{x\,dx}{\sec x} \Rightarrow \tan y = -\cos x - x\sin x + C$

4. $2x^2dx - 3\sqrt{y}\csc x\,dy = 0 \Rightarrow 3\sqrt{y}\,dy = \frac{2x^2}{\csc x}dx \Rightarrow 2y^{3/2} = 2(2-x^2)\cos x + 4x\sin x + C$
$\Rightarrow y^{3/2} = (2-x^2)\cos x + 2x\sin x + C_1$

5. $y' = \frac{e^y}{xy} \Rightarrow ye^{-y}dy = \frac{dx}{x} \Rightarrow (y+1)e^{-y} = -\ln|x| + C$

6. $y' = xe^{x-y}\csc y \Rightarrow y' = \frac{x\,e^x}{e^y}\csc y \Rightarrow \frac{e^y}{\csc y}dy = x\,e^x dx \Rightarrow \frac{e^y}{2}(\sin y - \cos y) = (x-1)e^x + C$

7. $x(x-1)dy - y\,dx = 0 \Rightarrow x(x-1)dy = y\,dx \Rightarrow \frac{dy}{y} = \frac{dx}{x(x-1)} \Rightarrow \ln y = \ln(x-1) - \ln(x) + C$
$\Rightarrow \ln y = \ln(x-1) - \ln(x) + \ln C_1 \Rightarrow \ln y = \ln\left(\frac{C_1(x-1)}{x}\right) \Rightarrow y = \frac{C_1(x-1)}{x}$

8. $y' = (y^2-1)(x^{-1}) \Rightarrow \frac{dy}{y^2-1} = \frac{dx}{x} \Rightarrow \frac{\ln\left(\frac{y-1}{y+1}\right)}{2} = \ln x + C \Rightarrow \ln\left(\frac{y-1}{y+1}\right) = 2\ln x + \ln C_1 \Rightarrow \frac{y-1}{y+1} = C_1x^2$

9. $2y' - y = xe^{x/2} \Rightarrow y' - \frac{1}{2}y = \frac{x}{2}e^{x/2}$.
$p(x) = -\frac{1}{2}$, $v(x) = e^{\int\left(-\frac{1}{2}\right)dx} = e^{-x/2}$.
$e^{-x/2}\,y' - \frac{1}{2}e^{-x/2}\,y = \left(e^{-x/2}\right)\left(\frac{x}{2}\right)\left(e^{x/2}\right) = \frac{x}{2} \Rightarrow \frac{d}{dx}\left(e^{-x/2}\,y\right) = \frac{x}{2} \Rightarrow e^{-x/2}\,y = \frac{x^2}{4} + C \Rightarrow y = e^{x/2}\left(\frac{x^2}{4} + C\right)$

10. $\frac{y'}{2} + y = e^{-x}\sin x \Rightarrow y' + 2y = 2e^{-x}\sin x$.
$p(x) = 2$, $v(x) = e^{\int 2dx} = e^{2x}$.
$e^{2x}y' + 2e^{2x}y = 2e^{2x}e^{-x}\sin x = 2e^x\sin x \Rightarrow \frac{d}{dx}(e^{2x}\,y) = 2e^x\sin x \Rightarrow e^{2x}\,y = e^x(\sin x - \cos x) + C$
$\Rightarrow y = e^{-x}(\sin x - \cos x) + Ce^{-2x}$

11. $xy' + 2y = 1 - x^{-1} \Rightarrow y' + \left(\frac{2}{x}\right)y = \frac{1}{x} - \frac{1}{x^2}$.
$v(x) = e^{2\int \frac{dx}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2$.
$x^2y' + 2xy = x - 1 \Rightarrow \frac{d}{dx}(x^2y) = x - 1 \Rightarrow x^2y = \frac{x^2}{2} - x + C \Rightarrow y = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$

12. $xy' - y = 2x\ln x \Rightarrow y' - \left(\frac{1}{x}\right)y = 2\ln x$.
$v(x) = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$. $\left(\frac{1}{x}\right)y' - \left(\frac{1}{x}\right)^2 y = \frac{2}{x}\ln x \Rightarrow$
$\frac{d}{dx}\left(\frac{1}{x} \cdot y\right) = \frac{2}{x}\ln x \Rightarrow \frac{1}{x} \cdot y = [\ln x]^2 + C \Rightarrow y = x[\ln x]^2 + Cx$

13. $(1 + e^x)dy + (ye^x + e^{-x})dx = 0 \Rightarrow (1 + e^x)y' + e^x y = -e^{-x} \Rightarrow y' = \frac{e^x}{1+e^x}y = \frac{-e^{-x}}{(1+e^x)}$.
$v(x) = e^{\int \frac{e^x dx}{(1+e^x)}} = e^{\ln(e^x+1)} = e^x + 1$.
$(e^x + 1)y' + (e^x + 1)\left(\frac{e^x}{1+e^x}\right)y = \frac{-e^{-x}}{(1+e^x)}(e^x + 1) \Rightarrow \frac{d}{dx}[(e^x + 1)y] = -e^{-x} \Rightarrow (e^x + 1)y = e^{-x} + C$
$\Rightarrow y = \frac{e^{-x}+C}{e^x+1} = \frac{e^{-x}+C}{1+e^x}$

14. $e^{-x}\,dy + (e^{-x}\,y - 4x)dx = 0 \Rightarrow \frac{dy}{dx} + y = 4x\,e^x \Rightarrow p(x) = 1,\ v(x) = e^{\int 1dx} = e^x \Rightarrow e^x\,\frac{dy}{dx} + y\,e^x = 4x\,e^{2x}$
$\Rightarrow \frac{d}{dx}(y\,e^x) = 4x\,e^{2x} \Rightarrow y\,e^x = \int 4x\,e^{2x}\,dx \Rightarrow y\,e^x = 2x\,e^{2x} - e^{2x} + C \Rightarrow y = 2x\,e^x - e^x + C\,e^{-x}$

15. $(x + 3y^2)\,dy + y\,dx = 0 \Rightarrow x\,dy + y\,dx = -3y^2dy \Rightarrow \frac{d}{dx}(xy) = -3y^2dy \Rightarrow xy = -y^3 + C$

16. $x\,dy + (3y - x^{-2}\cos x)\,dx = 0 \Rightarrow y' + \left(\frac{3}{x}\right)y = x^{-3}\cos x$. Let $v(y) = e^{\int \frac{3dx}{x}} = e^{3\ln x} = e^{\ln x^3} = x^3$.
Then $x^3\,y' + 3x^2\,y = \cos x$ and $x^3\,y = \int \cos x\,dx = \sin x + C$. So $y = x^{-3}(\sin x + C)$

17. $(x + 1)\frac{dy}{dx} + 2y = x \Rightarrow y' + \left(\frac{2}{x+1}\right)y = \frac{x}{x+1}$. Let $v(x) = e^{\int \frac{2}{x+1}dx} = e^{2\ln(x+1)} = e^{\ln(x+1)^2} = (x+1)^2$.
So $y'(x + 1)^2 + \frac{2}{(x+1)}(x + 1)^2 y = \frac{x}{(x+1)}(x + 1)^2 \Rightarrow \frac{d}{dx}\left[y(x + 1)^2\right] = x(x + 1) \Rightarrow y(x + 1)^2 = \int x(x + 1)dx$
$\Rightarrow y(x + 1)^2 = \frac{x^3}{3} + \frac{x^2}{2} + C \Rightarrow y = (x + 1)^{-2}\left(\frac{x^3}{3} + \frac{x^2}{2} + C\right)$. We have $y(0) = 1 \Rightarrow 1 = C$. So
$y = (x + 1)^{-2}\left(\frac{x^3}{3} + \frac{x^2}{2} + 1\right)$

18. $x\frac{dy}{dx} + 2y = x^2 + 1 \Rightarrow y' + \left(\frac{2}{x}\right)y = x + \frac{1}{x}$. Let $v(x) = e^{\int\left(\frac{2}{x}\right)dx} = e^{\ln x^2} = x^2$. So $x^2y' + 2xy = x^3 + x$
$\Rightarrow \frac{d}{dx}(x^2y) = x^3 + x \Rightarrow x^2y = \frac{x^4}{4} + \frac{x^2}{2} + C \Rightarrow y = \frac{x^2}{4} + \frac{C}{x^2} + \frac{1}{2}$. We have $y(1) = 1 \Rightarrow 1 = \frac{1}{4} + C + \frac{1}{2} \Rightarrow C = \frac{1}{4}$.
So $y = \frac{x^2}{4} + \frac{1}{4x^2} + \frac{1}{2} = \frac{x^4+2x^2+1}{4x^2}$

19. $\frac{dy}{dx} + 3x^2y = x^2$. Let $v(x) = e^{\int 3x^2dx} = e^{x^3}$. So $e^{x^3}y' + 3x^2e^{x^3}y = x^2e^{x^3} \Rightarrow \frac{d}{dx}\left(e^{x^3}y\right) = x^2e^{x^3} \Rightarrow e^{x^3}y = \frac{1}{3}e^{x^3} + C$.
We have $y(0) = -1 \Rightarrow e^{0^3}(-1) = \frac{1}{3}e^{0^3} + C \Rightarrow -1 = \frac{1}{3} + C \Rightarrow C = -\frac{4}{3}$ and $e^{x^3}y = \frac{1}{3}e^{x^3} - \frac{4}{3} \Rightarrow y = \frac{1}{3} - \frac{4}{3}e^{-x^3}$

20. $xdy + (y - \cos x)dx = 0 \Rightarrow xy' + y - \cos x = 0 \Rightarrow y' + \left(\frac{1}{x}\right)y = \frac{\cos x}{x}$. Let $v(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$.
So $xy' + x\left(\frac{1}{x}\right)y = \cos x \Rightarrow \frac{d}{dx}(xy) = \cos x \Rightarrow xy = \int \cos x\,dx \Rightarrow xy = \sin x + C$. We have $y\left(\frac{\pi}{2}\right) = 0 \Rightarrow \left(\frac{\pi}{2}\right)0 = 1 + C$
$\Rightarrow C = -1$. So $xy = -1 + \sin x \Rightarrow y = \frac{-1+\sin x}{x}$

21. $xy' + (x - 2)y = 3x^3e^{-x} \Rightarrow y' + \left(\frac{x-2}{x}\right)y = 3x^2e^{-x}$. Let $v(x) = e^{\int\left(\frac{x-2}{x}\right)dx} = e^{x-2\ln x} = \frac{e^x}{x^2}$. So
$\frac{e^x}{x^2}y' + \frac{e^x}{x^2}\left(\frac{x-2}{x}\right)y = 3 \Rightarrow \frac{d}{dx}\left(y \cdot \frac{e^x}{x^2}\right) = 3 \Rightarrow y \cdot \frac{e^x}{x^2} = 3x + C$. We have $y(1) = 0 \Rightarrow 0 = 3(1) + C \Rightarrow C = -3$
$\Rightarrow y \cdot \frac{e^x}{x^2} = 3x - 3 \Rightarrow y = x^2e^{-x}(3x - 3)$

22. $y\,dx + (3x - xy + 2)dy = 0 \Rightarrow \frac{dx}{dy} + \frac{3x - xy + 2}{y} = 0 \Rightarrow \frac{dx}{dy} + \frac{3x}{y} - x = -\frac{2}{y} \Rightarrow \frac{dx}{dy} + \left(\frac{3}{y} - 1\right)x = -\frac{2}{y}$.
$P(y) = \frac{3}{y} - 1 \Rightarrow \int P(y)dy = 3\ln y - y \Rightarrow v(y) = e^{3\ln y - y} = y^3e^{-y}$
$y^3e^{-y}x' + y^3e^{-y}\left(\frac{3}{y} - 1\right)x = -2y^2e^{-y} \Rightarrow y^3e^{-y}x = \int -2y^2e^{-y}dy = 2e^{-y}(y^2 + 2y + 2) + C$
$\Rightarrow y^3 = \frac{2(y^2 + 2y + 2) + Ce^y}{x}$. We have $y(2) = -1 \Rightarrow -1 = \frac{2(1 - 2 + 2) + Ce^{-1}}{2} \Rightarrow C = -4e$ and
$\Rightarrow y^3 = \frac{2(y^2 + 2y + 2) - 4e^{y+1}}{x}$

23. To find the approximate values let $y_n = y_{n-1} + (y_{n-1} + \cos x_{n-1})(0.1)$ with $x_0 = 0$, $y_0 = 0$, and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

x	y
0	0
0.1	0.1000
0.2	0.2095
0.3	0.3285
0.4	0.4568
0.5	0.5946
0.6	0.7418
0.7	0.8986
0.8	1.0649
0.9	1.2411
1.0	1.4273

x	y
1.1	1.6241
1.2	1.8319
1.3	2.0513
1.4	2.2832
1.5	2.5285
1.6	2.7884
1.7	3.0643
1.8	3.3579
1.9	3.6709
2.0	4.0057

24. To find the approximate values let $y_n = y_{n-1} + (2 - y_{n-1})(2x_{n-1} + 3)(0.1)$ with $x_0 = -3$, $y_0 = 1$, and 20 steps. Use a spreadsheet, graphing calculator, or CAS to obtain the values in the following table.

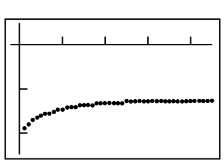
x	y
−3.0	1.0000
−2.9	0.7000
−2.8	0.3360
−2.7	−0.0966
−2.6	−0.5998
−2.5	−1.1718
−2.4	−1.8062
−2.3	−2.4913
−2.2	−3.2099
−2.1	−3.9393
−2.0	−4.6520

x	y
−1.9	−5.3172
−1.8	−5.9026
−1.7	−6.3768
−1.6	−6.7119
−1.5	−6.8861
−1.4	−6.8861
−1.3	−6.7084
−1.2	−6.3601
−1.1	−5.8585
−1.0	−5.2298

25. To estimate $y(3)$, let $y = y_{n-1} + \left(\frac{x_{n-1} - 2y_{n-1}}{x_{n-1} + 1}\right)(0.05)$ with initial values $x_0 = 0$, $y_0 = 1$, and 60 steps. Use a spreadsheet, graphing calculator, or CAS to obtain $y(3) \approx 0.8981$.

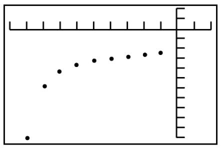
26. To estimate $y(4)$, let $z_n = y_{n-1} + \left(\frac{x_{n-1}^2 - 2y_{n-1} + 1}{x_{n-1}}\right)(0.05)$ with initial values $x_0 = 1$, $y_0 = 1$, and 60 steps. Use a spreadsheet, graphing calculator, or CAS to obtain $y(4) \approx 4.4974$.

27. Let $y_n = y_{n-1} + \left(\frac{1}{e^{x_{n-1} + y_{n-1} + 2}}\right)(dx)$ with starting values $x_0 = 0$ and $y_0 = 2$, and steps of 0.1 and −0.1. Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.
(a)



[−0.2, 4.5] by [−2.5, 0.5]

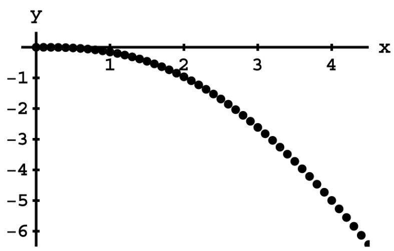
(b) Note that we choose a small interval of x-values because the y-values decrease very rapidly and our calculator cannot handle the calculations for $x \le -1$. (This occurs because the analytic solution is $y = -2 + \ln(2 - e^{-x})$, which has an asymptote at $x = -\ln 2 \approx 0.69$. Obviously, the Euler approximations are misleading for $x \le -0.7$.)



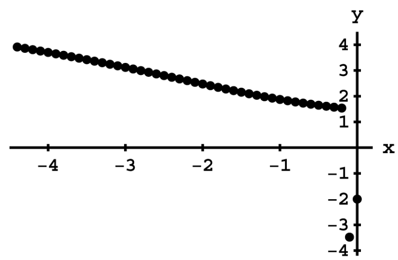
[−1, 0.2] by [−10, 2]

28. Let $y_n = y_{n-1} - \left(\frac{x_{n-1}^2 + y_{n-1}}{e^{y_{n-1}} + x_{n-1}}\right)(dx)$ with starting values $x_0 = 0$ and $y_0 = 0$, and steps of 0.1 and -0.1. Use a spreadsheet, programmable calculator, or CAS to generate the following graphs.

(a)



(b)



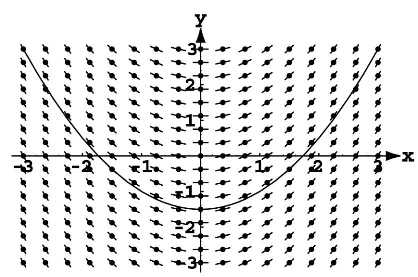
29.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−0.8	−0.56	−0.28	0.04	0.4

$\frac{dy}{dx} = x \Rightarrow dy = x\,dx \Rightarrow y = \frac{x^2}{2} + C$; $x = 1$ and $y = -1$

$\Rightarrow -1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \Rightarrow y(\text{exact}) = \frac{x^2}{2} - \frac{3}{2}$

$\Rightarrow y(2) = \frac{2^2}{2} - \frac{3}{2} = \frac{1}{2}$ is the exact value.



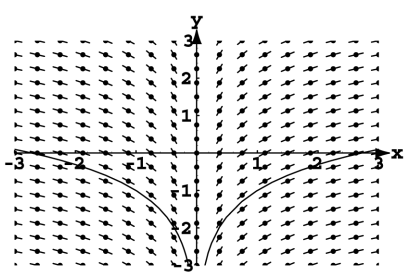
30.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−0.8	−0.6333	−0.4904	−0.3654	−0.2544

$\frac{dy}{dx} = \frac{1}{x} \Rightarrow dy = \frac{1}{x}dx \Rightarrow y = \ln|x| + C$; $x = 1$ and $y = -1$

$\Rightarrow -1 = \ln 1 + C \Rightarrow C = -1 \Rightarrow y(\text{exact}) = \ln|x| - 1$

$\Rightarrow y(2) = \ln 2 - 1 \approx -0.3069$ is the exact value.



31.

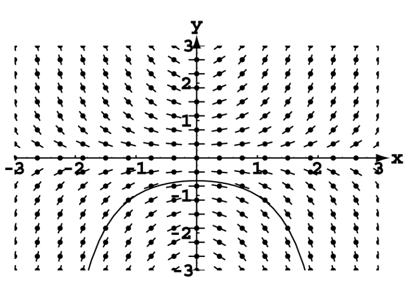
x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−1.2	−0.488	−1.9046	−2.5141	−3.4192

$\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C$

$\Rightarrow y = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} \cdot e^C = C_1 e^{\frac{x^2}{2}}$; $x = 1$ and $y = -1$

$\Rightarrow -1 = C_1 e^{1/2} \Rightarrow C_1 = -e^{1/2} y(\text{exact}) = -e^{1/2} \cdot e^{\frac{x^2}{2}}$

$= -e^{(x^2-1)/2} \Rightarrow y(2) = -e^{3/2} \approx -4.4817$ is the exact value.



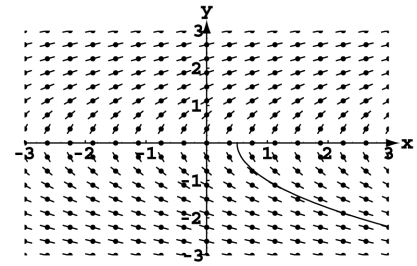
32.

x	1	1.2	1.4	1.6	1.8	2.0
y	−1	−1.2	−1.3667	−1.5130	−1.6452	−1.7688

$\frac{dy}{dx} = \frac{1}{y} \Rightarrow y\,dy = dx \Rightarrow \frac{y^2}{2} = x + C$; $x = 1$ and $y = -1$

$\frac{1}{2} = 1 + C \Rightarrow C = -\frac{1}{2} \Rightarrow y^2 = 2x - 1$

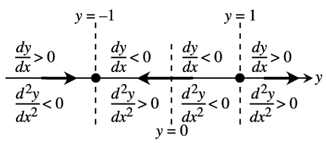
$\Rightarrow y(\text{exact}) = \sqrt{2x-1} \Rightarrow y(2) = -\sqrt{3} \approx -1.7321$ is the exact value.



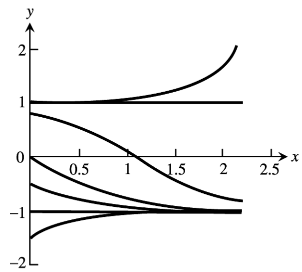
33. $\frac{dy}{dx} = y^2 - 1 \Rightarrow y' = (y+1)(y-1)$. We have $y' = 0 \Rightarrow (y+1) = 0, (y-1) = 0 \Rightarrow y = -1, 1$.

(a) Equilibrium points are -1 (stable) and 1 (unstable)

(b) $y' = y^2 - 1 \Rightarrow y'' = 2yy' \Rightarrow y'' = 2y(y^2-1) = 2y(y+1)(y-1)$. So $y'' = 0 \Rightarrow y = 0, y = -1, y = 1$.



(c)



34. $\frac{dy}{dx} = y - y^2 \Rightarrow y' = y(1-y)$. We have $y' = 0 \Rightarrow y(1-y) = 0 \Rightarrow y = 0, 1 - y = 0 \Rightarrow y = 0, 1$.

(a) The equilibrium points are 0 and 1. So, 0 is unstable and 1 is stable.

(b) Let $\longrightarrow$ = increasing, $\longleftarrow$ = decreasing.

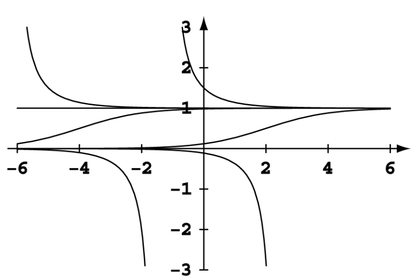
$y' < 0$ ← • 0 → $y' > 0$ • 1 ← $y' < 0$ → y

$y' = y - y^2 \Rightarrow y'' = y' - 2yy' \Rightarrow y'' = (y - y^2) - 2y(y - y^2) = y - y^2 - 2y^2 + 2y^3 \Rightarrow y'' = 2y^3 - 3y^2 + y$ $= y(2y^2 - 3y + 1) \Rightarrow y'' = y(2y-1)(y-1)$. So, $y'' = 0 \Rightarrow y = 0, 2y - 1 = 0, y - 1 = 0 \Rightarrow y = 0, y = \frac{1}{2}$, $y = 1$.

Let $\longrightarrow$ = concave up, $\longleftarrow$ = concave down.

$y'' < 0$ ← • 0 → $y'' > 0$ • 1/2 ← $y'' < 0$ • 1 → $y'' > 0$ → y

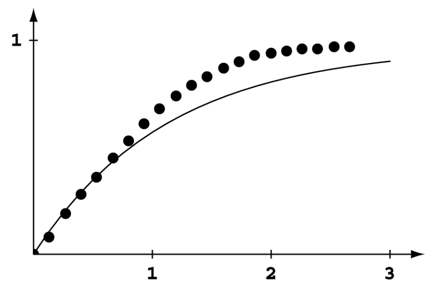
(c)



35. (a) Force = Mass times Acceleration (Newton's Second Law) or $F = ma$. Let $a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$. Then $ma = -mgR^2s^{-2} \Rightarrow a = -gR^2s^{-2} \Rightarrow v\frac{dv}{ds} = -gR^2s^{-2} \Rightarrow v\,dv = -gR^2s^{-2}ds \Rightarrow \int v\,dv = \int -gR^2s^{-2}ds$ $\Rightarrow \frac{v^2}{2} = \frac{gR^2}{s} + C_1 \Rightarrow v^2 = \frac{2gR^2}{s} + 2C_1 = \frac{2gR^2}{s} + C$. When $t = 0$, $v = v_0$ and $s = R \Rightarrow v_0^2 = \frac{2gR^2}{R} + C$ $\Rightarrow C = v_0^2 - 2gR \Rightarrow v^2 = \frac{2gR^2}{s} + v_0^2 - 2gR$

(b) If $v_0 = \sqrt{2gR}$, then $v^2 = \frac{2gR^2}{s} \Rightarrow v = \sqrt{\frac{2gR^2}{s}}$, since $v \ge 0$ if $v_0 \ge \sqrt{2gR}$. Then $\frac{ds}{dt} = \frac{\sqrt{2gR^2}}{\sqrt{s}} \Rightarrow \sqrt{s}\,ds = \sqrt{2gR^2}\,dt$
$\Rightarrow \int s^{1/2}ds = \int \sqrt{2gR^2}\,dt \Rightarrow \frac{2}{3}s^{3/2} = \sqrt{2gR^2}t + C_1 \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + C$; $t = 0$ and $s = R$
$\Rightarrow R^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)(0) + C \Rightarrow C = R^{3/2} \Rightarrow s^{3/2} = \left(\frac{3}{2}\sqrt{2gR^2}\right)t + R^{3/2} = \left(\frac{3}{2}R\sqrt{2g}\right)t + R^{3/2}$
$= R^{3/2}\left[\left(\frac{3}{2}R^{-1/2}\sqrt{2g}\right)t + 1\right] = R^{3/2}\left[\left(\frac{3\sqrt{2gR}}{2R}\right)t + 1\right] = R^{3/2}\left[\left(\frac{3v_0}{2R}\right)t + 1\right] \Rightarrow s = R\left[1 + \left(\frac{3v_0}{2R}\right)t\right]^{2/3}$

36. $\frac{v_0 m}{k}$ = coasting distance $\Rightarrow \frac{(0.86)(30.84)}{k} = 0.97 \Rightarrow k \approx 27.343$. $s(t) = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) \Rightarrow s(t) = 0.97\left(1 - e^{-(27.343/30.84)t}\right)$
$\Rightarrow s(t) = 0.97(1 - e^{-0.8866t})$. A graph of the model is shown superimposed on a graph of the data.



CHAPTER 9 ADDITIONAL AND ADVANCED EXERCISES

1. (a) $\frac{dy}{dt} = k\frac{A}{V}(c - y) \Rightarrow dy = -k\frac{A}{V}(y - c)dt \Rightarrow \frac{dy}{y - c} = -k\frac{A}{V}dt \Rightarrow \int \frac{dy}{y - c} = -\int k\frac{A}{V}dt \Rightarrow \ln|y - c| = -k\frac{A}{V}t + C_1$
$\Rightarrow y - c = \pm e^{C_1}e^{-k\frac{A}{V}t}$. Apply the initial condition, $y(0) = y_0 \Rightarrow y_0 = c + C \Rightarrow C = y_0 - c$
$\Rightarrow y = c + (y_0 - c)e^{-k\frac{A}{V}t}$.
(b) Steady state solution: $y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[c + (y_0 - c)e^{-k\frac{A}{V}t}\right] = c + (y_0 - c)(0) = c$

2. $\frac{d(mv)}{dt} = F + (v + u)\frac{dm}{dt} \Rightarrow F = \frac{d(mv)}{dt} - (v + u)\frac{dm}{dt} \Rightarrow F = m\frac{dv}{dt} + v\frac{dm}{dt} - v\frac{dm}{dt} - u\frac{dm}{dt} \Rightarrow F = m\frac{dv}{dt} - u\frac{dm}{dt}$.
$\frac{dm}{dt} = -b \Rightarrow m = -|b|t + C$. At $t = 0$, $m = m_0$, so $C = m_0$ and $m = m_0 - |b|t$.
Thus, $F = (m_0 - |b|t)\frac{dv}{dt} - u|b| = -(m_0 - |b|t)|g| \Rightarrow \frac{dv}{dt} = -g + \frac{u|b|}{m_0 - |b|t} \Rightarrow v = -gt - u\ln\left(\frac{m_0 - |b|t}{m_0}\right) + C_1$
$v = 0$ at $t = 0 \Rightarrow C_1 = 0$. So $v = -gt - u\ln\left(\frac{m_0 - |b|t}{m_0}\right) = \frac{dy}{dt} \Rightarrow y = \int\left[-gt - u\ln\left(\frac{m_0 - |b|t}{m_0}\right)\right]dt$ and $u = c$, $y = 0$ at
$t = 0 \Rightarrow y = -\frac{1}{2}gt^2 + c\left[t + \left(\frac{m_0 - |b|t}{|b|}\right)\ln\left(\frac{m_0 - |b|t}{m_0}\right)\right]$

3. (a) Let y be any function such that $v(x)y = \int v(x)Q(x)\,dx + C$, $v(x) = e^{\int P(x)\,dx}$. Then
$\frac{d}{dx}(v(x)\cdot y) = v(x)\cdot y' + y\cdot v'(x) = v(x)Q(x)$. We have $v(x) = e^{\int P(x)\,dx} \Rightarrow v'(x) = \ = e^{\int P(x)\,dx}P(x) = v(x)P(x)$.
Thus $v(x)\cdot y' + y\cdot v(x)\,P(x) = v(x)Q(x) \Rightarrow y' + y\,P(x) = Q(x) \Rightarrow$ the given y is a solution.
(b) If v and Q are continuous on $[a, b]$ and $x \in (a, b)$, then $\frac{d}{dx}\left[\int_{x_0}^{x} v(t)Q(t)\,dt\right] = v(x)Q(x)$
$\Rightarrow \int_{x_0}^{x} v(t)Q(t)\,dt = \int v(x)Q(x)\,dx$. So $C = y_0 v(x_0) - \int v(x)Q(x)\,dx$. From part (a), $v(x)y = \int v(x)Q(x)\,dx + C$.
Substituting for C: $v(x)y = \int v(x)Q(x)\,dx + y_0 v(x_0) - \int v(x)Q(x)\,dx \Rightarrow v(x)y = y_0 v(x_0)$ when $x = x_0$.

4. (a) $y' + P(x)y = 0$, $y(x_0) = 0$. Use $v(x) = e^{\int P(x)\,dx}$ as an integrating factor. Then $\frac{d}{dx}(v(x)y) = 0 \Rightarrow v(x)y = C$
$\Rightarrow y = Ce^{-\int P(x)\,dx}$ and $y_1 = C_1e^{-\int P(x)\,dx}$, $y_2 = C_2e^{-\int P(x)\,dx}$, $y_1(x_0) = y_2(x_0) = 0$, $y_1 - y_2 = (C_1 - C_2)e^{-\int P(x)\,dx}$
$= C_3e^{-\int P(x)\,dx}$ and $y_1 - y_2 = 0 - 0 = 0$. So $y_1 - y_2$ is a solution to $y' + P(x)y = 0$ with $y(x_0) = 0$.
(b) $\frac{d}{dx}(v(x)[y_1(x) - y_2(x)]) = \frac{d}{dx}\left(e^{\int P(x)\,dx}\left[e^{-\int P(x)\,dx}(C_1 - C_2)\right]\right) = \frac{d}{dx}(C_1 - C_2) = \frac{d}{dx}(C_3) = 0$.
$\int \frac{d}{dx}(v(x)[y_1(x) - y_2(x)])dx = (v(x)[y_1(x) - y_2(x)]) = \int 0\,dx = C$

(c) $y_1 = C_1 e^{-\int P(x)\,dx}$, $y_2 = C_2 e^{-\int P(x)\,dx}$, $y = y_1 - y_2$. So $y(x_0) = 0 \Rightarrow C_1 e^{-\int P(x)\,dx} - C_2 e^{-\int P(x)\,dx} = 0$
$\Rightarrow C_1 - C_2 = 0 \Rightarrow C_1 = C_2 \Rightarrow y_1(x) = y_2(x)$ for $a < x < b$.

5. $(x^2 + y^2)dx + x\,y\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-(x^2+y^2)}{x\,y} = -\frac{x}{y} - \frac{y}{x} = -\frac{1}{y/x} - \frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = -\frac{1}{v} - v \Rightarrow \frac{dx}{x} + \frac{dv}{v - F(v)} = 0$
$\Rightarrow \frac{dx}{x} + \frac{dv}{v - \left(-\frac{1}{v} - v\right)} = 0 \Rightarrow \int \frac{dx}{x} + \int \frac{v\,dv}{2v^2+1} = C \Rightarrow \ln|x| + \frac{1}{4}\ln|2v^2+1| = C \Rightarrow 4\ln|x| + \ln\left|2\left(\frac{y}{x}\right)^2 + 1\right| = C$
$\Rightarrow \ln|x^4| + \ln\left|\frac{2y^2+x^2}{x^2}\right| = C \Rightarrow \ln\left|x^2(2y^2 + x^2)\right| = C \Rightarrow x^2(2y^2 + x^2) = e^C \Rightarrow x^2(2y^2 + x^2) = C$

6. $x^2\,dy + (y^2 - x\,y)dx = 0 \Rightarrow \frac{dy}{dx} = \frac{-(y^2 - x\,y)}{x^2} \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^2 + \frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = -v^2 + v \Rightarrow \frac{dx}{x} + \frac{dv}{v - (-v^2 + v)} = 0$
$\Rightarrow \int \frac{dx}{x} + \int \frac{dv}{v^2} = C \Rightarrow \ln|x| - \frac{1}{v} = C \Rightarrow \ln|x| - \frac{1}{y/x} = C \Rightarrow \ln|x| - \frac{x}{y} = C$

7. $\left(x\,e^{y/x} + y\right)dx - x\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{x\,e^{y/x} + y}{x} = e^{y/x} + \frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = e^v + v \Rightarrow \frac{dx}{x} + \frac{dv}{v - (e^v + v)} = 0$
$\Rightarrow \int \frac{dx}{x} - \int \frac{dv}{e^v} = C \Rightarrow \ln|x| + e^{-v} = C \Rightarrow \ln|x| + e^{-y/x} = C$

8. $(x + y)dy + (x - y)dx = 0 \Rightarrow \frac{dy}{dx} = \frac{-(x-y)}{x+y} = \frac{\frac{y}{x} - 1}{1 + \frac{y}{x}} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = \frac{v-1}{1+v} \Rightarrow \frac{dx}{x} + \frac{dv}{v - \left(\frac{v-1}{1+v}\right)} = 0$
$\Rightarrow \int \frac{dx}{x} + \int \frac{(1+v)dv}{v^2+1} = 0 \Rightarrow \int \frac{dx}{x} + \int \frac{dv}{v^2+1} + \int \frac{v\,dv}{v^2+1} = 0 \Rightarrow \ln|x| + \tan^{-1}v + \frac{1}{2}\ln|v^2+1| = C$
$\Rightarrow 2\ln|x| + 2\tan^{-1}v + \ln\left|\left(\frac{y}{x}\right)^2 + 1\right| = C \Rightarrow \ln|x^2| + 2\tan^{-1}\left(\frac{y}{x}\right) + \ln\left|\frac{y^2+x^2}{x^2}\right| = C \Rightarrow 2\tan^{-1}\left(\frac{y}{x}\right) + \ln\left|y^2 + x^2\right| = C$

9. $y' = \frac{y}{x} + \cos\left(\frac{y-x}{x}\right) = \frac{y}{x} + \cos\left(\frac{y}{x} - 1\right) = F\left(\frac{y}{x}\right) \Rightarrow F(v) = v + \cos(v-1) \Rightarrow \frac{dx}{x} + \frac{dv}{v - (v + \cos(v-1))} = 0$
$\Rightarrow \int \frac{dx}{x} - \int \sec(v-1)\,dv = 0 \Rightarrow \ln|x| - \ln|\sec(v-1) + \tan(v-1)| = C \Rightarrow \ln|x| - \ln\left|\sec\left(\frac{y}{x} - 1\right) + \tan\left(\frac{y}{x} - 1\right)\right| = C$

10. $\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{-\left(x\sin\frac{y}{x} - y\cos\frac{y}{x}\right)}{x\cos\frac{y}{x}} = \frac{y}{x} - \tan\frac{y}{x} = F\left(\frac{y}{x}\right) \Rightarrow F(v) = v - \tan v$
$\Rightarrow \frac{dx}{x} + \frac{dv}{v - (v - \tan v)} = 0 \Rightarrow \int \frac{dx}{x} + \int \cot v\,dv = 0 \Rightarrow \ln|x| + \ln|\sin v| = C \Rightarrow \ln|x| + \ln\left|\sin\frac{y}{x}\right| = C$